Formalizing Cut Elimination of Coalgebraic Logics in Coq

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Summary

- in Coq, formalize ⅔ of

Cut Elimination in Coalgebraic Logics

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Abstract

We give two generic proofs for cut elimination in propositional modal logics, interpreted over coalgebras. We first investigate semantic coherence conditions between the axiomatisation of a particular logic and its coalgebraic semantics that guarantee that the cut-rule is admissi-

- formalisation of syntax, semantics and 2 cut-elimination theorems for (generic) propositional multi-modal logic
- **K** as example, (work in progress on coalition logic)
- revealed only 4 errors (which were easy to correct)
- see `http://askra.de/science/coalgebraic-cut`

Motivation

Verified Cut Elimination

- Cut elimination is an important meta property of a logic
- ... but is tricky to prove
- ... and proofs are rarely ever spelled out

Generic Nature of Coalgebraic Modal Logics

- results apply to every logic that fits into the framework
- formalising the preconditions suffices
  to obtain formalised soundness, completeness and cut-elimination results

This work is the basis for

- certified validity checkers extracted from the completeness proof

Cut Elimination

Semantic: Given a proof for Γ

- soundness shows validity of Γ
- cut-free completeness shows the existence of a cut-free proof

Syntactic: Shift cut upwards, replacing, for instance,

$$\dfrac{(\neg\wedge)\,\dfrac{\vdash \neg A, \neg B, C}{\vdash \neg(A \wedge B), C} \qquad \dfrac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\,(\wedge)}{\vdash C}\,(\text{cut})$$

by

$$\dfrac{(\text{cut})\,\dfrac{\vdash \neg A, \neg B, C \qquad \vdash A}{\vdash \neg B, A} \qquad \vdash B}{\vdash C}\,(\text{cut})$$

Outline

- Introduction
- **Formalization in Coq**
  - syntax
  - proofs
  - ~~semantics~~
- **Selection of Major Results**
- **Some Interesting Bits**
  - classical vs. intuitionistic logic
  - 1 of the 4 problems found during the formalisation
- **Conclusion**

Coalgebraic Modal Logics: Formulas

Multi-modal Propositional Modal Logic

- parametric on modal similarity type Λ
  which provides the set of modal operators and their arity
- formulas: p, $f \wedge g$, $\neg f$, $\heartsuit(f_1, \ldots, f_n)$
  for some set of propositional variables V, $p \in V$ and $\heartsuit$ of arity n

```
Record modal_operators : Type := { operator : Type; arity : operator → nat }.
Variable (V : Type) (L : modal_operators).

Inductive lambda_formula : Type :=
  | lf_prop : V → lambda_formula
  | lf_neg : lambda_formula → lambda_formula
  | lf_and : lambda_formula → lambda_formula → lambda_formula
  | lf_modal : forall(op : operator L),
      counted_list lambda_formula (arity L op) → lambda_formula.
```

- counted_list A n are lists over A of length n

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Coalgebraic Modal Logics: Rules I

Fixed Propositional Rules

$$\frac{}{\vdash \Gamma, p, \neg p}\,(\mathsf{Ax}) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,(\wedge) \qquad \frac{\vdash \Gamma, \neg A, \neg B}{\vdash \Gamma, \neg(A \wedge B)}\,(\neg\wedge)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \neg\neg A}\,(\neg\neg) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\,(\mathsf{cut})$$

Definition sequent : **Type** := list lambda_formula. *(* modulo reordering *)*
Record sequent_rule : **Type** := {assumptions: list sequent; conclusion: sequent}.

Coalgebraic Modal Logics: Rules I

Fixed Propositional Rules

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Coalgebraic Modal Logics: Rules II

Logic Specific 1-Step Rules for Modalities

$$\frac{\vdash a_1^1, \ldots, \neg b_1^1, \ldots \qquad \cdots \qquad \vdash a_1^k, \ldots, \neg b_1^k, \ldots}{\vdash \heartsuit_1(\ldots), \ldots, \neg\heartsuit_1'(\ldots), \ldots}$$

Subject to Additional Conditions

- non-empty conclusion
- arguments for the modal operators in the conclusion are unnegated propositional variables
- all variables in the assumptions appear in the conclusion
- proofs may contain substitution instances of 1-step rules

Coalgebraic Modal Logics: Proofs

Proofs are finite trees build from rules and assumptions

```
Inductive proof(rules : set sequent_rule)(hypotheses : set sequent)
                                              : sequent → Type :=
  | assume : forall(gamma : sequent),
      hypotheses gamma → proof rules hypotheses gamma
  | rule : forall(r : sequent_rule), rules r →
      dep_list sequent (proof rules hypotheses) (assumptions r) →
        proof rules hypotheses (conclusion r).
```

- proof R H G is the type of proof trees for sequent G using rules R and hypotheses H
- dep_list A T $[a_1; \ldots; a_n]$ is a inhomogeneous list of n elements where the i-th element has type T a_i
- very concise formalisation relying on dependent types

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Formalized Results

```
Variable T : functor.

Lemma cut_free_completeness :
  forall(enum_V : enumerator V)(LS : lambda_structure)
        (rules : set sequent_rule)(osr : one_step_rule_set rules)(s : sequent),
    classical_logic →
    non_trivial_functor T →
    one_step_cut_free_complete (enum_elem enum_V) LS rules osr →
    valid_all_models (enum_elem enum_V) LS s →
      provable (GR_set rules) empty_sequent_set s.
```

Formalized Results II

```
Variable op_eq : eq_type (operator L).
Variable v_eq : eq_type V.

Theorem syntactic_admissible_cut :
  forall(rules : set sequent_rule),
    countably_infinite V →
    one_step_rule_set rules →
    absorbs_congruence rules →
    absorbs_contraction op_eq v_eq rules →
    absorbs_cut op_eq v_eq rules →
      admissible_rule_set (GR_set rules) empty_sequent_set is_cut_rule.
```

Application to K

using the rule set $\dfrac{\vdash \neg p_1, \ldots, \neg p_n, p_0}{\vdash \neg\Box p_1, \ldots \neg\Box p_n, \Box p_0}$

```
Theorem k_semantic_cut :
  classical_logic →
    admissible_rule_set (GR_set k_rules) (empty_sequent_set VN KL) is_cut_rule.

Theorem k_syntactic_cut :
  admissible_rule_set (GR_set k_rules) (empty_sequent_set VN KL) is_cut_rule.

Lemma k_nd_equiv : forall(s : sequent VN KL),
  provable (GRC_set k_rules) (empty_sequent_set VN KL) s ↔
    provable (GRC_set is_k_n_rule) k_d_axioms s.
```

Outline

Introduction

Formalization in Coq

Selection of Major Results

Some Interesting Bits

Conclusion

Classical vs. Intuitionistic Logic

Classical object logic of Pattinson & Schröder

- rules $\dfrac{}{\vdash \Gamma, p, \neg p}$ (Ax) and $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \neg\neg A}$ $(\neg\neg)$
- defined disjunction: $A \vee B \stackrel{\text{def}}{=} \neg(\neg A \wedge \neg B)$

Coq's intuitionistic meta logic

- $A \vee \neg A$ is not a tautology, but $\neg(\neg A \wedge \neg\neg A)$ is
- $\neg\neg A \rightarrow A$ is not a tautology, but $A \rightarrow \neg\neg A$ is

Expect, that some results of Pattinson & Schröder are not provable in Coq

- making Coq classical: **Require** Classical.
- I prefer

Definition classical_logic : **Prop** := **forall**(P : **Prop**), ¬ ¬ P → P.

The need for classical reasoning

. . . depends on disjunction and the semantic of sequents

- disjunction is syntactic sugar: $A \lor B \stackrel{\text{def}}{=} \neg(\neg A \land \neg B)$ in the object logic
- semantic of sequents ($[\![-]\!]_S$) is defined via the semantic of formulas ($[\![-]\!]_F$)

$$[\![\Gamma]\!]_S \stackrel{\text{def}}{=} [\![\bigvee \Gamma]\!]_F$$

$$[\![A, B]\!]_S \stackrel{\text{def}}{=} [\![A \lor B]\!]_F = [\![\neg(\neg A \land \neg B)]\!]_F$$

Double negation translation has surprising effects

- $\dfrac{}{\vdash \Gamma, p, \neg p}$ (Ax) is sound, because $\neg(\neg p \land \neg\neg p)$ is tautological
- $\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}$ (cut) is only sound when assuming classical_logic,

because $A \land \neg(\neg B \land \neg\neg A) \to B$ is not a tautology

Substitution Lemma

Lemma (original substitution lemma)

Assume

- Γ *is provable with rules of modal rank n (i.e.,* Γ *has rank n)*
- σ *is a substitution that maps to formulas of modal rank k*

Then $\Gamma\sigma$ *is provable with rules of modal rank* $n + k$*,*
using the additional assumptions Ax_k*, where*

$$\mathrm{Ax}_k \stackrel{\mathrm{def}}{=} \{\Gamma, A, \neg A \mid \Gamma \text{ and } A \text{ of modal rank } k\}$$

Proof.

Take the original proof, substituting $\neg p\sigma, p\sigma, \Gamma$ from Ax_k for $\dfrac{}{\vdash \Gamma, p, \neg p}$ (Ax)

□

Wrong Substitution Lemma

Lemma (original substitution lemma)

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Example

- $\Gamma = \heartsuit(p), p, \neg p$ of modal rank $n = 1$, provable by (Ax)
- $\sigma : p \mapsto \heartsuit(p)$ of modal rank $k = 1$
- but $\Gamma\sigma = \heartsuit(\heartsuit(p)), \heartsuit(p), \neg\heartsuit(p)$ of rank $n + k = 2$
  is not in Ax_1

Substitution Lemma II

Error seems to break the main theorems

- subst. lemma is used inside induction proofs on the modal rank
- Γ of rank 1, σ of rank k
- reduces $\Gamma\sigma$ of rank $k+1$ to Ax_k of rank k
- thus permitting the use of the induction hypothesis

Use $\mathrm{Ax}_\sigma^{n+k} = \{\Gamma, p\sigma, \neg p\sigma \mid \Gamma \text{ of modal rank } n+k\}$

- "binding" of σ makes other proofs simpler
- need to use weakening before applying the induction hypothesis
- this way, original proofs remain valid

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Conclusion I

Summary

- soundness, completeness, cut-elimination results for generic multi-modal propositional logic in Coq
- modal logic **K** as example
- very concise formalisation of syntax, semantics, proofs relying on dependent types (without predicates for well-formedness)
- only 4 non-trivial problems revealed (+1 for coalition logic)
- the usual peer-review process does not ensure correctness

Future Work

- coalition logic (work in progress) and other example logics
- remaining content of the paper, especially interpolation theorem and interpolants
- change formalisation to extract certified tautology checkers

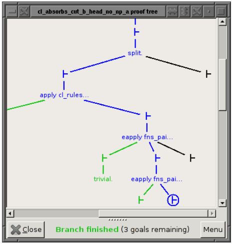
Conclusion II

Complexity

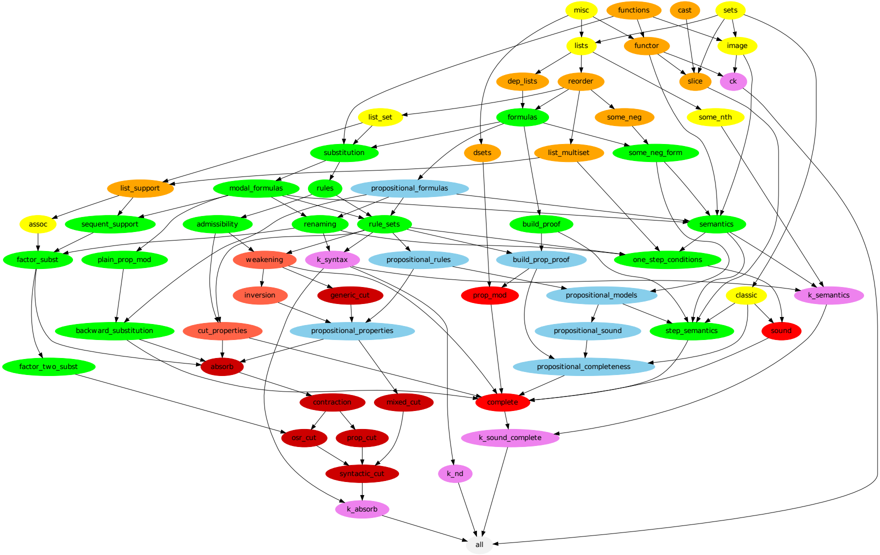
- 36,000 lines, 400 definitions, 1300 theorems in Coq
- for 19 propositions, 7 definitions, 3 examples on ≈ 31 pages

Side Effects

- parallel library compilation for Coq in Proof General
- proof tree visualisation



File Dependencies



Coalgebraic Modal Logics: Semantics

- a functor T describes the type of frames
- behaviour of modal operators is given by (fibred) predicate liftings: $[\![\heartsuit]\!] : \big((P_1 \subseteq X), \ldots, (P_n \subseteq X)\big) \mapsto (Q \subseteq TX)$
- a frame (model) is given by a coalgebra $\gamma : X \longrightarrow TX$ together with a valuation $\tau : V \longrightarrow \mathcal{P}(X)$
- formula semantics yields a subset of the state space $[\![-]\!]^c_\tau \subseteq X$:

$$\begin{aligned}
[\![p]\!]^c_\tau &= \tau(p) \\
[\![A \wedge B]\!]^c_\tau &= [\![A]\!]^c_\tau \cap [\![B]\!]^c_\tau \\
[\![\neg A]\!]^c_\tau &= X \setminus [\![A]\!]^c_\tau \\
[\![\heartsuit(A_1, \ldots, A_n)]\!]^c_\tau &= \gamma^{-1}\big([\![\heartsuit]\!]([\![A_1]\!]^c_\tau, \ldots, [\![A_n]\!]^c_\tau)\big)
\end{aligned}$$